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Variability in the Acquisition of Number in Preschool Children: An Approach from the Cardinal Number and the Notation of Quantities

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Abstract

This study followed a process-based approach, in which variability describe the changes that occurs moment-to-moment. The aim of this study is to explore the intra-individual trajectories of the joint use of the numerical cardinal and the notation of quantities. Twenty-three preschool children of 4-5 years old participated on the study. The children were attending public school in Cali-Colombia. Regarding the method, a microgenetic design and a task called "Give and Note a Number" was used to capture the variability of the children's performance. The task was an adaptation of the task "Give a Number" by LeCorre and Carey (2007). The technique of cluster analysis was implemented for data analysis. The results revealed some patterns of the intra-individual trajectories of the children's performance: (1) five patterns of cardinality, (2) five patterns of notations, and (3) several joint profiles to"give" and "note" quantities. In conclusion, our findings suggest a dynamic process of the joint use of the cardinality and notations, showing the co-existence of different levels of complexity.

Keywords: cardinality, intra-individual variability, preschool children, microgenetic design, notations.

1 Introduction

For decades, the development of the cardinal number and the notation of quantities have been the main topics of interest in mathematical cognition and early mathematics instruction studies (Baroody & Wilkins, 1999; Treacy & Wills, 2002). For instance, there is broad literature aboutthese aspects in early childhood: the development of cardinality and their implications on later mathematical skills (Cañellas & Rasetto, 2013; Geary, 2006). The development of numerical notations, the classification of production of written numerals, and its relation to the comprehension of the numerical system. Although both notions, cardinality, and notation of quantitieshave been deeply studied, in practice, there is a lack of information about their possible relationship (Kolkman, Kroesbergen & Leseman,2013). However, it is possible to infer a relationship between cardinality and the notation of quantities. Different approaches have shown the relevance of counting in the development of numerical skills. For instance, (1) the mathematical cognition approach suggests that children who manage to manage the count have the ability to operate with the base ten numbering system (Dowker, 2008; Gillian & Lewis, 1993; Saxton & Cakir, 2006). (2) From a numerical processing approach, it is proposed that the use of accurate counting with fingers facilitates children'sunderstanding of arithmetic skills. (3) The numerical cognition approach suggests that children's knowledge for quantifying quantities isinnate and therefore it will support the later acquisition of numerical knowledge in the scholarage (Geary & Moore, 2016)

In general, there have been several approaches to studying mathematical cognition. Some of the studies have pointed out the relevance of the "counting" procedures for the development of numerical skills arguing that procedures of quantification are essential for the understanding of the numerical system in base 10. For instance, Dowker (2008), claims that the cardinal principle of cardinal is an important predictor of children's performance of arithmetic tasks. In addition, Saxton, and Cakir (2006) suggested that counting training leads children to improve their performance in applied tasks about equivalence and decomposition numerical especially in the representation of numerical concepts while using blocks of value.

Another group of studies has addressed the relationship between counting and arithmetic skills (Lafay, Thevenot, Castel & Fayol, 2013; Crollen, Serón & Noël, 2011). These studies refer to the counting technique of "finger counting" as an external aid used by children to represent the numbers and to maintain the numerical sequence. The strategy of “finger counting” supports the performance of basic arithmetic operations and the transition in the development of non-symbolic and symbolic numerical abilities. In addition, Bafalluy and Noël (2008), indicate that the use of the fingers to quantify elements, allows children to support the processes of acquisition of numerical knowledge. The procedure of "finger counting" allows children to begin the representation of the numerical sequence by using the principle of 1:1 correspondence and stable order. In the same perspective, Costa et al. (2011) propose that the use of "finger counting" allow children to perform arithmetic tasks, as a mechanism that reduces the cognitive demand in working memory.

Finally, some studies have shown the relationship among analog representations of magnitude, understanding of the numerical cardinal, and writing and reading of Arabic numerals (Chu, VanMarle& Geary, 2015). For instance, within this approach, Geary and VanMarle, (2018) suggested that the mathematical achievement of children in formal schooling is based on their knowledge of cardinality. Libertus, Feigenson, and

Halberda (2013) indicates that there is a relationship between the numerical skills of counting and the analog system of magnitude. However, little is known about this relationship.

The purpose of this study is to explore the relationship between cardinality and quantity notation from the perspective of cognitive development. The study adopts a process-oriented approach, in which the interest is to observe the changes in a phenomenon over time, instead of using a static point of view. Therefore, a microgenetic design of three observation sessions) is used to capture the variability of children's performance by analyzing the intra-individual trajectories and patterns of children’s strategies of cardinality and notations of set size from 1 to 10. An adapted version of the task “Give a Number” (LeCorre & Carey, 2007) called “Give and Note a Number” is used. By using the adapted task, we examine how preschool children (4 to 5 years) put into practice the joint use of strategies for cardinality and their respective notation over the solution of a task. Specifically, the research question that addressed this study is: How are the trajectories of variability of the acquisition of the number developed in the joint use of both the cardinal number and the notation of quantities in preschool children (4- to -5 years)?

The aims of this study are (1) identifying the trajectories of variability in the use of the cardinal number in a group of preschool children when solving a task that demands giving X cardinal number, (2) identifying the trajectories of variability in the use of the notation of quantities in preschool children when solving a task that demands to represent X numerical quantity that has been previously counted, and (3) establishing possible patterns of variability in the joint use of the cardinal number and the notation of quantities in a group of preschool children.

2. Theoretical Background

2.1 The Development of Cardinality

Cardinality is an early and core principle for later acquisition of mathematical skills. For this reason, the study of its development has played an important role in the comprehension of early mathematical cognition. In this regard, Gelman & Gallistel (1992) have provided relevant evidence about the nature and development of cardinality as a keystone of counting skills. Gelman (1978) proposed five basic principles that guide the learning of counting. Two of them give rise to the understanding of ordinalities, such as the 1:1 correspondence, where each item of a collection is assigned a numerical label, and the stable order, in which the numerical labels that are used must maintain an ordered sequence. The other three principles give rise to the understanding of the cardinal number, such as cardinality, in which the last label assigned in the count represents the total of items in a set; the abstraction, in which any object can be counted independently of its characteristics; and the irrelevant order, in which the numerical cardinal is obtained independently of the order of the counting process.

According to the literature, the acquisition of the principle of cardinality is relevant to determine that an individual has learned to count (Bermejo, Morales & deOsuna, 2004; LeCorre & Carey, 2007; Sarnecka & Lee, 2009). Therefore, children who master the numerical cardinal establish a relationship between their initial representations of the number and their cultural knowledge of the numerical sequence. Regarding the construction of cardinality, Wynn (1990) proposes two types of representation: (1) cardinal label, which refers to the ability to mentally represent the cardinal number without pronouncing the number word in the counting activity; and (2) cardinal word, which indicates that the last word pronounced in the count inactivity represents the numerosity of a set of elements. These two forms of representation are related to quantification procedures such as subitization (cardinal label) and count (cardinal word), as possible ways to determine the cardinal number (Dehaene, 1992; Piazza, Mechelli, Butterworth & Price, 2002).

Studies on cardinality have focused on approaching the discussion from different conceptions of cognitive development. One of them has focused on a conception of development in which the advances in knowledge are associated with the chronological age, by using cross-sectional and longitudinal approaches (Escudero, Rodríguez, Lago & Enesco, 2014; LeCorre & Carey, 2007, 2008; Posid & Cordes, 2015; Rodríguez, Martí & Salsa, 2018). From this perspective, some findings stand out. For instance, Wynn (1990) finds a relationship between age and the type of representation for the numerical cardinal. In this way, 2.6-year-olds are considered "grabbers" when showing low success to give the cardinal, whereas 3.6-year-olds are classified as "counters" because they show high success to give the cardinal. The results obtained by Wynn (1992) show that at an early age (between 2 and 3years) children, before counting, understand the number of words. On the other hand, Sarnecka andCarey (2008) show that 2 and 4-year-olds, before understanding the cardinal word principle, respond procedurally in counting situations.

Another perspective addressed for the studies on cardinality, from the microgenetic method, assumes that the development processes show modifications in short periods of time. As a result,the microgenetic method reveals different paths that cannot be observed from a linear approach todevelopment. The microgenetic method allows capturing the intra- and inter-individual variability in individuals' performance (Bermejo et al., 2004;Blote, Otterloo, Stevenson & Veenman, 2004; Fischer & Bidell, 1998). In a four-session microgenetic study, Bermejo (2005) found that children assigned to an experimental group acquirethe cardinal number in a few days from exposure to the multiple trials of a problem when it normally takes several months to achieve this numerical knowledge. On the other hand, Chetland and Fluck(2007), in a microgenetic study with five waves ofdata collection, used a cardinality task called "Give X". The authors found that children between 2.2 and 4.8 years use different types of strategies that show the use of cardinality, such as: "grabber", "counter", "taker" and "combination" strategy (counter-grabber or counter-taker). In addition, the results indicated that many children who used the "grabber"–"taker" strategy, performed well by indicating the last word that was counted. These results show that the understanding of cardinality does not lie only in the use of verbal strategies.

Finally, the findings of three studies are described due to their findings contribute to theunderstanding of the principle of cardinality at an early age: In a study by Gibson, Gunderson, and Levine (2020), a group of children between 3 and4 years old was asked to work accompanied withtheir parents to read books with numerical content along four weeks. The results revealed that the interaction around numerical knowledge affects their understanding of the numerical cardinal. A study by Paliwal and Baroody (2018) with a group of children between 2 to 5 years old consisted on carried out four training sessions and six experimental training sessions (5 weeks), in one of the following conditions: label, and then count, emphasize, and repeat the last word, and counting only. The findings show that the best way to promote the acquisition of the cardinality principle is to carry out activities with small sets, inwhich the children are given the word number. Therefore, the children can subitize the quantity and easily access understanding from the numeric cardinal. O'Rear & McNeil (2019) carriedout a study with a group of 4-year-old children, consisting of six sessions over a period of six weeks. The children were assigned to one this conditions: 1) an experimental "count and label practice condition", 2) an experimental "label-first condition" and 3) "control condition". The study shows that proposing set subitizing activities before counting is important because this favors the understandingof cardinality.

In general terms, studies on the understanding of cardinality from traditional methodological approaches (cross-sectional and longitudinal) describe the acquisition of cardinality as the presence or absence of formal symbolic knowledge of what the quantity represents, while the results of microgenetic studies show that this understanding can be presented gradually (i.e., before using verbal strategies). The findings about cardinality in early ages reveal that their development is mediated by contextual or pedagogical variables that facilitate the acquisition of this counting principle.

2.2 Notational Development

The interpretation and production of written numerals play an important role in the mathematical domain, due to these skills are relevant from the beginning of the schooling process. Conceptually, the notation of quantities requires the use of principles that guide their learning and the way in which "alphabetic" notation differs from the "Arabic" notation (Brizuela & Cayton, 2010; Lee, Karmiloff- Smith, Cameron & Dodsworth, 1998). The study of notational development has been conceived as a stage-like process according to chronological age. Munn (2001) proposes that the acquisition ofthe symbolic function begins with the use of iconic symbols characterized by the correspondence of the numerical value (e.g., drawings, paintings, or images). Later, the children advance in their knowledge by using conventional symbols (i.e., "Arabic" notations) to represent the numerical quantity. In this regard, Tolchinsky, and Karmiloff-Smith (1993) identify different levels of representation related to age: 4-year-olds express the numerical concepts using the completesyntagm "two wheels" and they denote the quantity using drawings. In contrast, 6-year-olds use the complete syntagm "five wheels" and to denote the quantity they allude to the Arabic number system "5". Finally, Scheuer, De la Cruz, and Iparraguirre (2010), find that children have different levels of development for each type of representation. For the drawing, the children begin to use non-representational strokes until they reach very clear drawings. For numbers, children begin with representations based onshapes for numbers until they properly use a conventional notation (i.e., "Arabic" notation). These findings suggest that the development of numerical notation is a complex process that requires different moments until using the "Arabic" numerical system under a communicative function.

The literature on numerical knowledge also has been interested in the production of notations (see

Lerner & Sadovsky, 1994; Sinclair & Scheuer, 1993). Scheuer, Sinclair, de Rivas & Tièche (2000) establish seven categories of analysis from which it is possible to classify the notations: 1) conventional numerical notations; which refer to the correct Arabic notation; 2), multiple notations that correspond to the representation of elements in a set, (e.g., checkmarks, pseudo letters and/or numbers). 3) forms for numbers, corresponding to a graph that does not represent quantity (e.g., “r”); 4) form for the sort of numbers, representing a quantity inrelation to their characteristics (e.g., to note “one hundred”, children draw an image of large dimension). 5) Logographic notations, represent the notations based on the decomposition of the word number to note the quantity (e.g., note “206” for “twenty-six”). 6) Compacted notations are based on the number enunciated (e.g., “one hundred seventy-one”, 10071). 7) other notations that do not correspond to any of the categories mentioned above. In a study by Scheuer, Santamaria, and Echenique (2016), it was found that 4-year-old children, frequently used multiple notations rather than logographic notations. This happened when children were asked for defined quantities (i.e., number series or quantification of manipulable objects). However, the children used many logogrammic notations when they were asked for indefinite quantities (i.e.,expression of the absence of quantity or imagination and expression of large numbers).

2.3 Variability and Microgenetic Method

Studies on cognitive development have shown the presence of variability as a constant in processes at the macro level (development) and micro level (learning). In this regard, van Dijk, and van Geert (2014) point out that in the last two decades, one of the interests of developmental psychology has focused on the study of processes, observing their change as they occur. In other words, the focus of interest is not only on identifying which behaviors change at a given time but on giving an account of how these changes emerge and develop over time. This recent approach to developmental studies has brought with its methodological challenges such as establishing repeated measures to observe and follow the variability in short- and long-term processes. The microgenetic method (Siegler & Crowley, 1991; Siegler, 1994), is characterized by (1) observations that cover the period of change, 2) a high density of observations and (3) moment-by-moment intensive analysis to infer the processes that give rise to quantitative and qualitative changes (Siegler & Crowley, 1991; p.606), this analysis has become one of the strategic methodological resources to give an accountof variability. In this regard, variability is no longer considered a methodological error, as it usedto be described in classical perspectives, but is considered a fundamental characteristic of the cognitive nature. Variability is present not only amongst individuals (inter-individual variability) when solving a task, but also in the individual itself (intra-individual variability) when solving the same situation on repeated occasions (e.g., Morra, Gobbo, Zopito & Sheese, 2008; Nesselroade & Salthouse, 2004; Rabbitt, Osman, Moore, & Stollery (2001).

Van Geert and van Dijk (2002) define intra-individual variability as "behavioral differences within the same individuals, at different times (van Geert & van Dijk, 2002, p.341). From a dynamic perspective of the developmental study, understood as an everchanging system, variability constitutes the very nature of development, besides constituting an indicator of transitions and changes that take place in it (see Thelen & Smith, 1994). Empirical studies have consistently shown that drastic increases in variability often precede moments of transition in developmental trajectories (e.g., Bassano & van Geert, 2007; Hayes, Laurenceau, Feldman, Strauss, and Cardaciotto, 2007; Vallacher, 2002). Therefore, critical fluctuations can be used to identify transition points of the system that is being studied, and thus establish which factors are involved in such change. In this sense, the decrease or increase of the variability or fluctuations is the result of the dynamics of the emergency and changes between patterns or regularities in the developmental trajectories which give rise to moments of stability or transitions in development (Thelen & Smith, 1994).

3. Method

To track the variability in the understanding of the cardinal number and the acquisition of the Arabic notation system to communicate quantities, it is proposed a microgenetic study that is characterized by extended observations which are analyzed intensively (Siegler, 2007; Siegler & Crowley, 1991). The microgenetic design consists of three observation sessions using the "Give and Note a Number" task. The sessions are one week apart between applications.

3. 1. Participants

Twenty-three 4- to 5-year-olds took part in the study, they attended a public school in Cali-Colombia. The criteria to include them in the study were the following: 1) children who were between 4 to 5 years old and attended preschool and 2) children who successfully demonstrate knowledge of the 1:1 correspondence principle and stable order are selected through a task called "Elicitation of the Count List".

3. 2. Instruments and Procedure

The task "Elicitation of the Count List" proposed by the researchers LeCorre & Carey (2007) has the purpose of guaranteeing that children know the principle of 1:1 correspondence and the principle of stable order. The task "Elicitation of the Count List" consists of a set of 10 plastic toys, the same for the set (e.g., 10 skateboards), and a glove puppet named "Jorge", (manipulated by the researcher) who presents the task to the child. The glove puppet presents the task and gives the child directions about what has to be done, that is, counting. The researcher presents the instruction for the task as follows: "Jorge [the glove puppet] is going to orderten toys on a table. As Jorge cannot count, he willask you for help to count the toys."

This procedure was carried out in a game session, individually, with an average time of 2 to 4 minutes per participant. In the game session, the amount of 10 objects was ordered by the puppet ina row. The task proposes the following criteria for the selection of participants: 1) the children who, when counting, succeed in the first or second trial, complete the task; and 2) the children who, when counting, make errors in the first two trials, make the third trial with the help of the experimenter, whopoints the toy, and the child says the number. In this task, the child who presents two errors is not selected. For each application, there was a printed registration grid, and the experimenter filled it in atthe time of applying the task.

3.3. Experimental task "Give and Note a Number"

The task "Give and Note a Number" is an adaptation made by Martinez, (2018), from the original task "Give a number", which was proposed by the researchers LeCorre & Carey (2007). The task maintains the same cognitive demand as "Give a number", but it has some variations: 1) the set size is extended from 6 to 10. 2) the part of the notation is added; 3) the strategies are observed instead of determining the level of knowledge, and 4) the conditions are differentiated between the cardinal principle and the cardinal word.

In this study, the implementation of the task corresponds to a naturalistic approach to children's performance that consists of the sequence of actions counting and writing numbers (notations). This procedure has been used in previous studies (see Scheuer et. al, 2000;Tolchinsky & Karmiloff Smith, 1993). Additionally, the procedure of counting to write notations is also the most common practice introduced in educational settings. Preschool children usually are provided with elements to count and later they are asked to write a corresponding notation.

The "Give and Note a Number" task consists of three sets of plastic toys, 10 same objects in each set (e.g., 10 skateboards, 10 cubes, 10 balls), pencil and paper, and a glove puppet named "Jorge", (manipulated by the researcher) that presents the task to the child. The glove puppet[1] presents the task and gives the child directions about the actions to be performed: 1) give X quantity and 2) note the quantity given. This procedure is repeated several times, from quantity 1 to quantity 10.

The researcher presents the instruction for the task as follows: "Jorge [the glove puppet] is going to place a transparent box that contains ten toys on the table. As Jorge cannot count, he will ask you for help so that you can give him the requested number of toys that are in the box. " For instance, Jorge tells the child “Please, give X skateboards". After the child gives the quantity of toys, Jorge [the glove puppet] says: "Please, can you note on this sheet the number of toys that you gave me?" This procedure was carried out in three game sessions, individually. The task does not have time restrictions. Solving the problem task demands from the children approximately 15 to 20 minutes for a set size of 1 to 10 elements. In each game session, the quantities of objects were presented by the glove puppet in ascending order, starting at 1 and ending at 10. The task does not have time restrictions, nor does it use the "success" criterion to continue with the presentation of quantities in the task. All game sessions were videotaped. According to the task protocol, 30 performance measures were obtained per child. The task was presented repeatedly during 3 sessions. Each session was presented one week apart for a total of three weeks.

3. 4. Categories of Analysis

Strategies proposed by Chetland and Fluck (2007): 1) grabber, the child randomly grabs a quantity of objects; 2) counter, the child says the word number as he picks up the objects; 3) taker, the child takes the objects silently and keeps the sequence; and 4) combination, the child combines verbal and non- verbal counting when giving the objects.

Sorts of notation proposed by Scheuer et al. (2000): 1) Analogy, the child draws circles or check marks when representing a numerical quantity, maintaining the principle of 1:1 correspondence. 2) Mixed, the child uses different sorts of notation to represent a numerical quantity; 3) Alphabetical, the child alludes to letters of the alphabet to represent a quantity; 4) pre-arabic, the child uses mirror notations to represent a numerical quantity; and 5) Arabic, the child performs the notation in a conventional way to represent a quantity.

The reliability was calculated from the double coding of 17% of the videos (12 out of 69 videos). The double coding was carried out for all the selected videos, which had an average length of 20 minutes. The percentage of agreement between coders was 84%, being in a "satisfactory" range with a Kappa value of 0.736 (see Fleiss, 1981).

3. 5. Data Analysis

To perform the data analysis, the strategies used by the participants to give an account of the numerical cardinal and the type of notation to which they refer to represent the numerical quantity are taken as reference. According to this, the following analyzes were carried out.

3.5.1. General analysis of performances to "Give" and "Note" quantities.

A descriptive analysis was carried out by using statistic software (IBM SPSS Static version 23) through which the percentage of "strategies to give quantities" and the percentage of "notations to communicate quantities" were established. This procedure was carried out in three sets size (1-3, 4-6, and 7-10).

3.5.2. Cluster analysis of performances to "Give" and "Note" quantities

From the trajectories of the individual performances of the children, a cluster analysis was carried out using the data mining software called "Tanagra"[2]. The cluster analysis allows establishing groups of cases for the children’s

[1] The glove puppet is used to establish empathy between the researcher and the child during the task.

performance to "give" and "note" quantities (trajectories of the 23 participants) according to their level of similarity. Specifically, the K-means cluster technique was used. The algorithm used in K-Means consists of assigning recurrently each data point to a K group[3]. Therefore, the description of each cluster refers not only to the group of children (profiles) but also to the characteristics of intra-individual trajectories involved in the cluster. To perform the cluster analysis, the most predominant strategy forcardinality and quantity notation obtained throughout the three sessions is selected.

3.5.3. Joint profiles of cardinality-notation based on the children's performance to "Give" and"Note Quantities".

From the individual performances of the children to "give quantities" and "note quantities", a descriptive statistical analysis was carried out through statistical software (IBM SPSS Statistic version 23). This procedure was carried out in three sets size (1-3, 4-6, and 7-10).

4. Results

According to our research questions, first, we introduced a general statistical analysis of the children's performance for the use of strategies and notations and later, introduced an analysis of the variability.

4.1. General statistical Analysis

Regarding the children's strategies to "give", Table 1 shows that there were significant relationships between the types of strategies to "give" and the achievement as well as the relationship between set size and the level of achievement in the use of strategies to "give" and the relationship between the children's strategies to "give" and the set size. These results indicate that children's performance for all set size was not due to randomness. Complementarily, the differences between set size and the use of strategies to "give" was significant only for the set size 1-3 and 7-10, but not for the numerical ranks of 1-3 and 4-6, neither to the numerical ranks4-6 and 7-10. (See Table 1).

Regarding the children's performance to note quantities, Table 2 shows that there were significant relationships between the types of notations and the level of achievement, the relationship between the types of notations to represent quantities (i.e., analogy, mixed, alphabetical, pre-arabic, and arabic) and the set size (i.e., 1-3, 4-6, 7-10), as well as for the relationshipbetween the children's notations and the set size. These results indicate that children's notation for each set size was not due to randomness. Complementarily, the differences between the use of notation in the set size were significant for the numerical ranks 1-3, and 7-10, but not for the set size of 1-3 and 4-6, nor for the set size 4-6 and7-10 (See Table 2).

4. 2. Variability of Children's Performance to "Give" and "Note" Quantities

The children's performances to "give" quantities, reveal two patterns in relation to the increase of the set size: 1) A decreasing pattern was observed for the percentage of use of the strategies "taker" and "grabber", 2) An increasing pattern was observedin the percentage of use of the strategies "counter"and "combination" (See Figure 1).

In relation to the children's performance to "note", was possible to identify a stable pattern in the percentage of use of notations along with all the set size). In particular, the arabic notation was highly used,along all set size. In contrast, the other types of notations were not so frequent (i.e., Alphabetic, pre-arabic, and Analogy) or did not appear in the children's performance (i.e., Mix notation), (See Figure 2).

[2] http://eric.univ-lyon2.fr/~ricco/tanagra/en/tanagra.html

[3] See definition in: https://www.datascience.com/blog/k-means-clustering

Table 1. Statistical Analysis of the Children's Strategies to "give"

Relation between Variables	*Types of strategies to "give"* (i.e., grabber, counter, taker, and combined) and *the achievement on the task*	*Achievement in the use of strategies to "give" and numerical rank* and the	*Strategies to "give" and the numerical ranks*	Differences between Strategies to "give" *and the numerical ranks*
Test	Chi-square test for independence	Chi-square test for independence	Friedman test	Wilcoxon test
Results	$X^2_{(3)} = 28.999$, *p*-value =0.000*	$X^2_{(2)} = 31.575$, *p*-value =0.000*	$X^2_{(2)} = 6.598$, *p*-value =0.037*	*Ranks 1-3 and 7-10* (Z= -3.635, *p*-value=0.000) * *Ranks 1-3 and 4-6* (Z= -1.802, *p*-value = 0.72) *Ranks 4-6 and 7-10* (Z= -1.709, *p*-value=0.82).

Note: * Significant p value

Table 2. Statistical Analysis of the Children's Notations

Relation between Variables	*Types of notations and the level of achievement*	*Types of notations and the numerical ranks*	*Notation and the numerical ranks*	Differences between Notation and the numerical ranks
Test	Chi-square test for independence	Chi-square test for independence	Friedman test	Wilcoxon test
Results	$X^2_{(3)} = 170.796$, *p*-value =0.000*	$X^2_{(2)} = 17.462$, *p*-value =0.000*	$X^2_{(2)} = 11.953$, *p*-value =0.003*	*Ranks 1-3 and 7-10* (Z= -2.524, *p* -value = 0.12) * *Ranks 1-3 and 4-6* (Z= -1.773, *p*-value =0.076) *Ranks 4-6 and 7-10* (Z= -1.009, *p*-value=0.313).

Note: * Significant p value

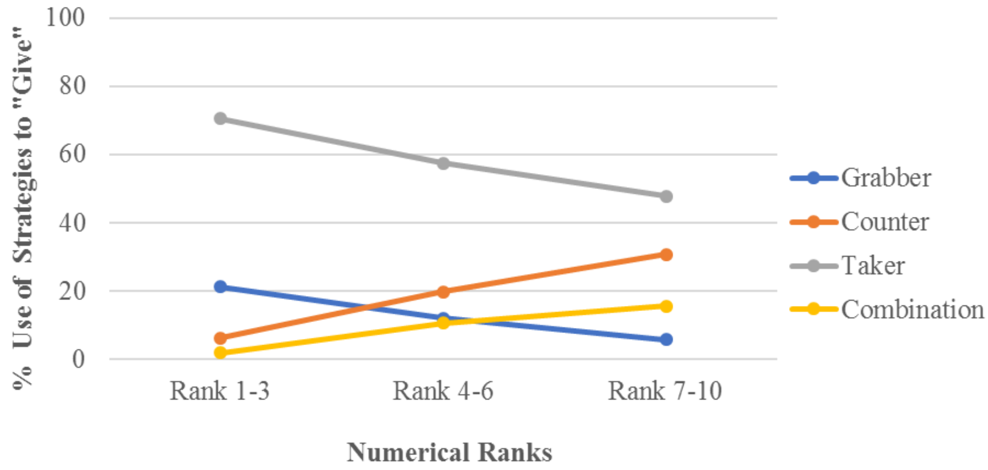

Figure 1. Variabily of the children’s performances to “give” quantities.

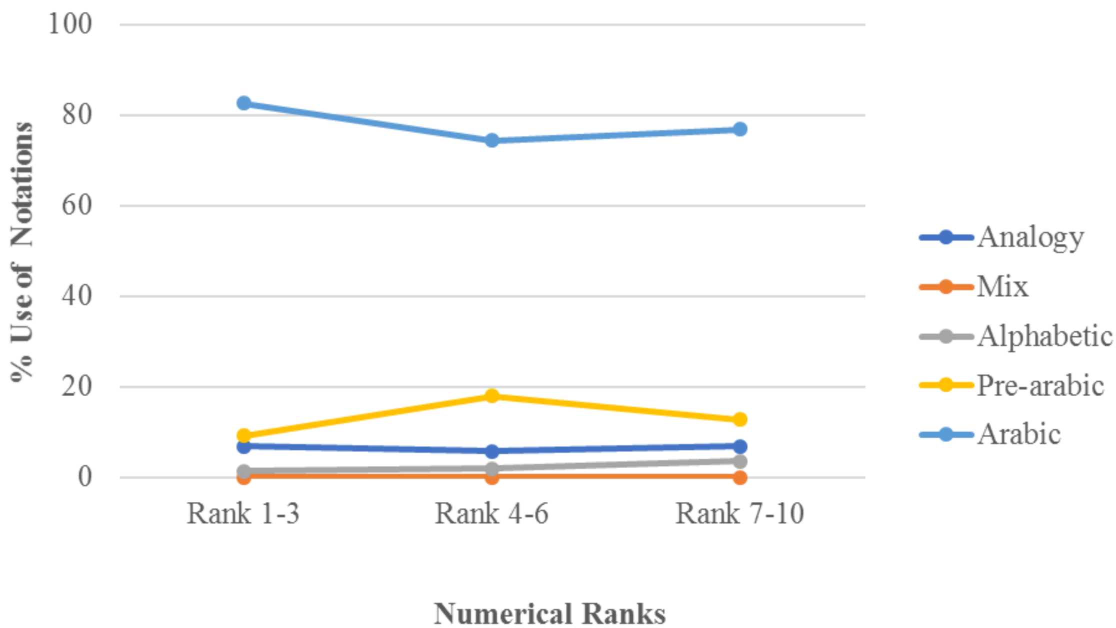

Figure 2. Variability of children´s performance to “note” quantitites.

4. 3. Similarities of the Children's Performance to "Give" and "Note" quantities from 1 to 10.

A clustering analysis (K-means) used to group children's strategies to "give", revealed five clusters. In Figure 3a, Cluster 1 (n= 9; 39.1% of the cases) shows children combining the strategies "grabber" (score 1) and "taker" (score 3), for all the numerical quantities. Figure 3b, Cluster 2 (n= 6; 26.1% of the cases), shows children's usingthe following four strategies: "grabber" (score 1), "counter" (score 2), "taker" (score 3), and "combination" (score 4), for all the numerical quantities. Figure 3c, Cluster3 (n= 3; 13% of the cases), shows children's combining two to three strategies to "give" such as "counter (score 3, "taker" (score 2), and "grabber" (score 1) for all the numerical quantities. Figure 3d, Cluster 4 (n=2; 8.7% of the cases), shows children's using a diversity of strategies to give quantities from 1 to 5 and later stabilize with the use of the strategy "taker" (score 2), from the quantities from 6 to 10. Finally, Figure 3e, Cluster 5 (n= 3; 13% of the cases), shows children who exclusively were using the strategy "taker" (score 3) along all the numerical quantities (See Figure 3).

In addition, the clustering analysis (K-means) was used to group children's "notations", resulting in five clusters. In Figure 4a, Cluster 1 (n=6; 26.1% of the cases), shows children using arabic notations from the quantities 1 to 3, and using "arabic notations" (score 5) and "pre-arabic" notations (score 4) for the numerical quantities from 4 to 10. In Figure 4b, Cluster 2 (n= 4; 17.4%), shows children using the "arabic notation" in most of the numerical quantities, combined in a less extend to the use of "pre- arabic" notation. In Figure 4c, Cluster 3 (n=4; 17.4%), shows children using "arabic" notations (score 5), (1 to 4, and 7 to 10), and "pre-arabic" notation (score 4). Figure 4d, Cluster 4 (n=7; 30.4%), shows children whose notation was stable along all the numerical quantities. Note that five of them used the most complex notation ("arabic" - score 5), while other children, used the less complex notation ("analogy" notation - score 1). Finally, Figure 4e, Cluster 5 (n= 2; 8.7%), shows two children that did not reveal a regularity among their notations (See Figure 4).

4.4. Patterns on Variability: The Joint Profiles of Cardinality-Notations

For each set size (1-3, 4-6, 7-10), we characterized the achievement profile of the 23 children's performance in terms of the joint use of strategies to "give" and to "note" quantities. This qualitative procedure revealed that the total achievement varied through the three set sizes. In the set size from 1 to 3, six achievement profiles were identified (See Figure 5). The two most frequent were the profiles "taker strategy- arabic notation" (69.6%) and "grabber strategy - arabic notation" (13%). For the numerical rank 4 to 6, nine achievement profiles were identified. The most frequent were the profiles "taker strategy-Arabic notation" (39.1%), and "counter strategy- arabic notation" (13%) (See Figure 6). Finally, for the set size 7 to 10, seven achievement profiles were identified. The most frequent were the profiles "counter strategy – Arabic notation" (30,4%), and "taker strategy – Arabic notation" (26.1%) (See Figure 7). In addition, for the set size 4-6 and 7-10, ideal profiles and error profiles were observed.

In summary, (a) the total joint profiles of cardinality-notation increased with the set size, (b) the number of joint profiles for achievement remains similar among the three-set size (6, 7, and 6), and (c) the joint profile "Taker - Arabic" appears and remain in all the set size.

5. Discussion

The present study aimed to explore the relationship between the counting strategies and notations used by the children to solve the task "Give and Note a Number". From a process approach, the microgenetic method allowed us to unhide thevariability as an informative source of the children's performances in a moment-to-moment analysis.

A common aspect of the studies on cardinality and notation has been its linear approach, focusing on the children's transitions from "non-knowers" to "knowers". For instance, with respect to cardinality studies, Wynn (1990) considers that

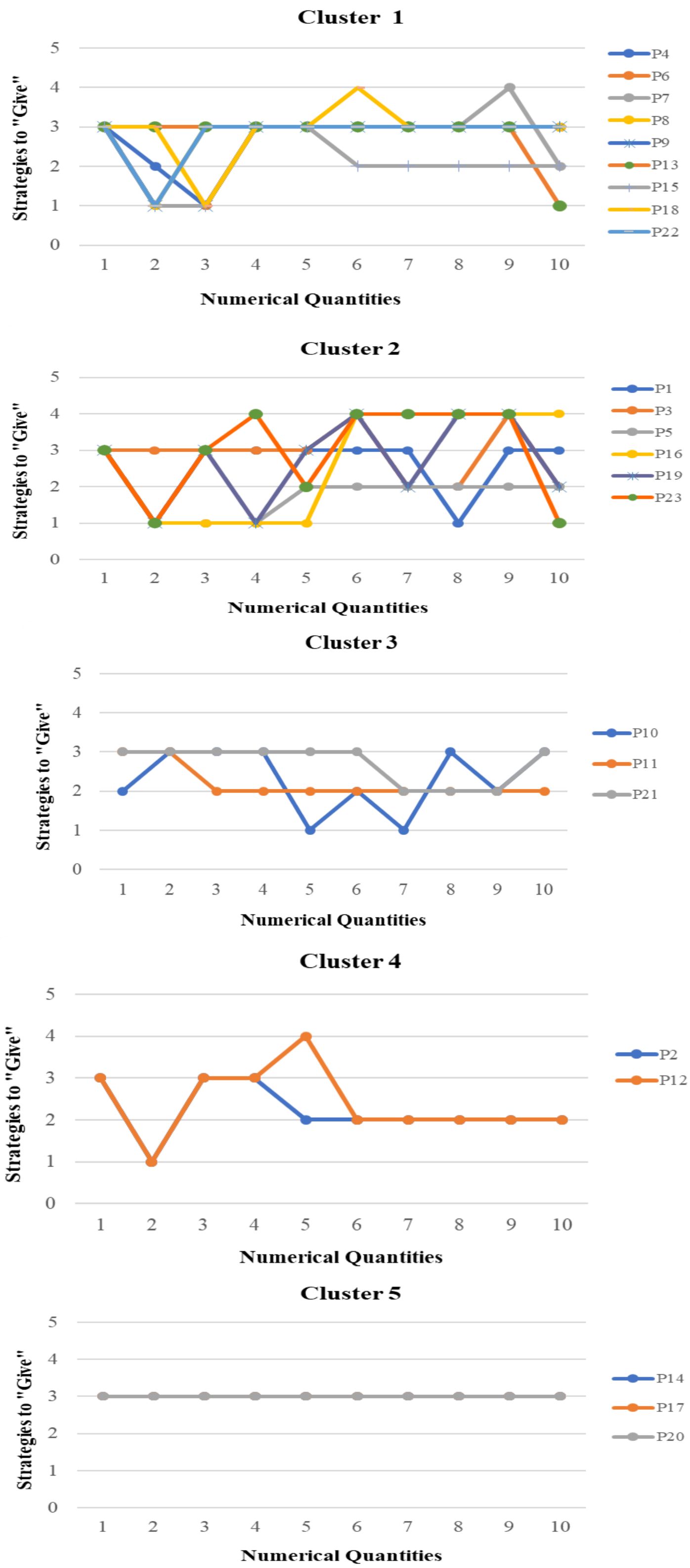

Figure 3. Cluster of Trajectories regarding the children's performance to "give" quantities. The y-axes correspond to the categories of strategies to "give": 1= "grabber", 2= "taker", 3= "counter", and 4= "combination".

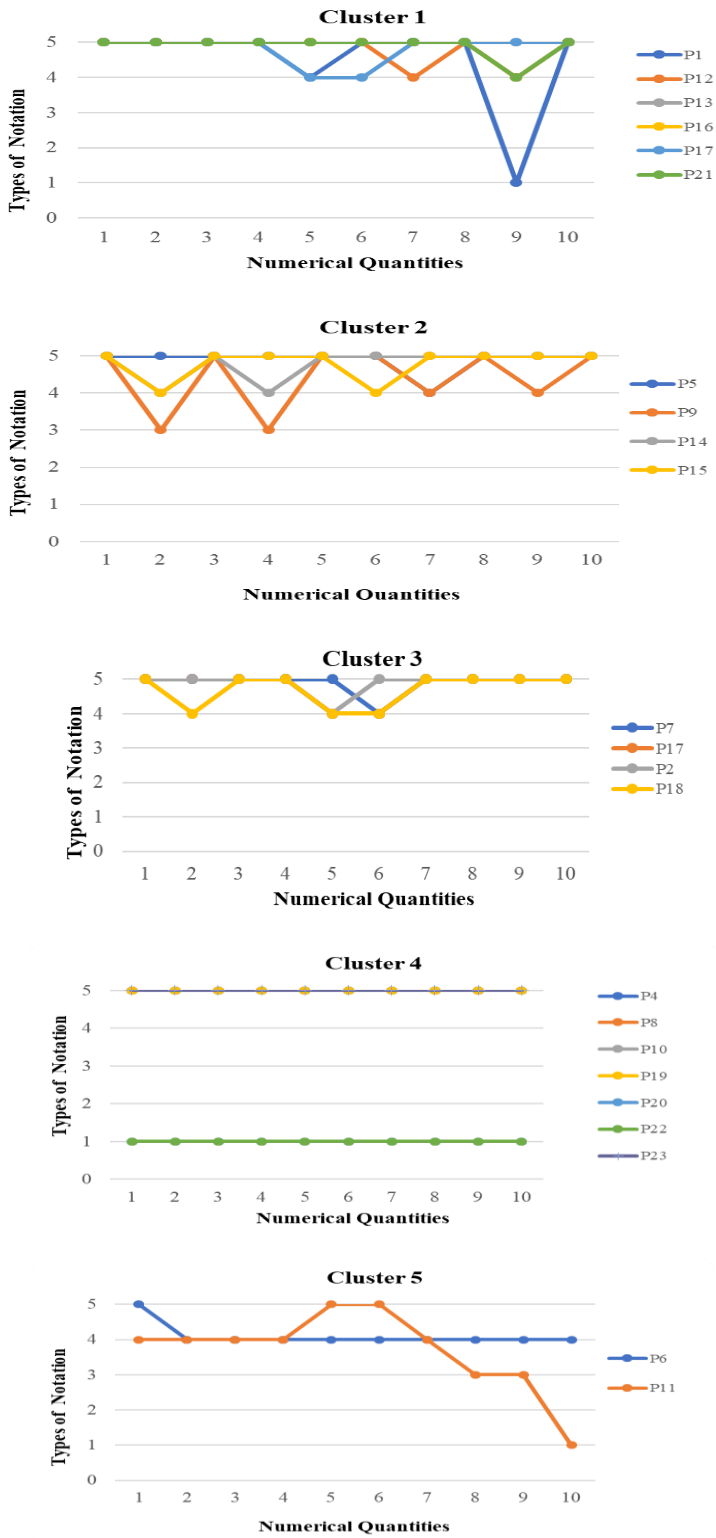

Figure 4. Cluster of Trajectories regarding the children's performance to "note" quantities. The y-axes correspond to the categories to "note": 1= "Analogy", 2= "Mix", 3= "Alphabetic", 4= "Pre-Arabic", and 5= "Arabic".

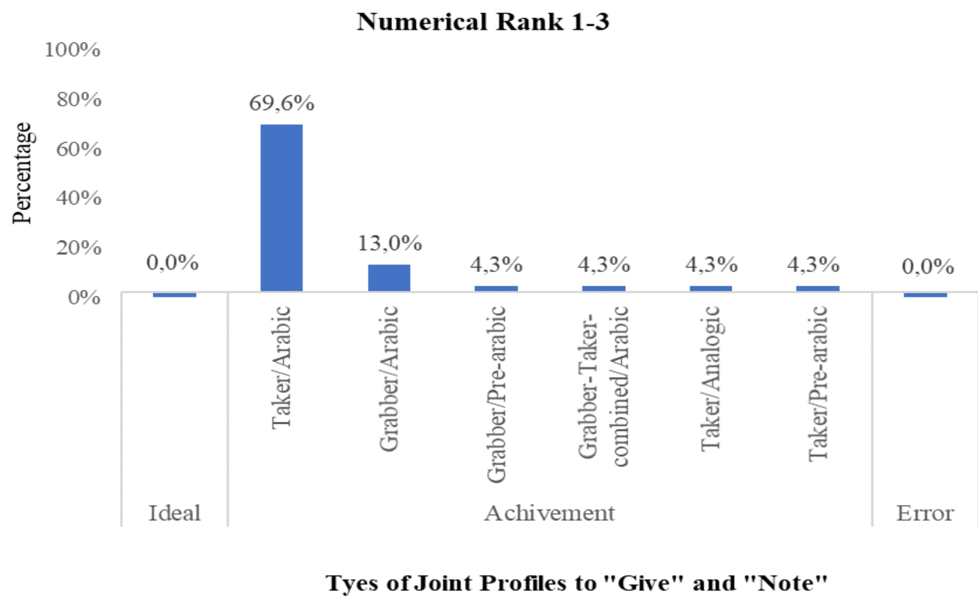

Figure 5. Percentage of joint profiles of cardinality-notation for the numerical rank 1-3.

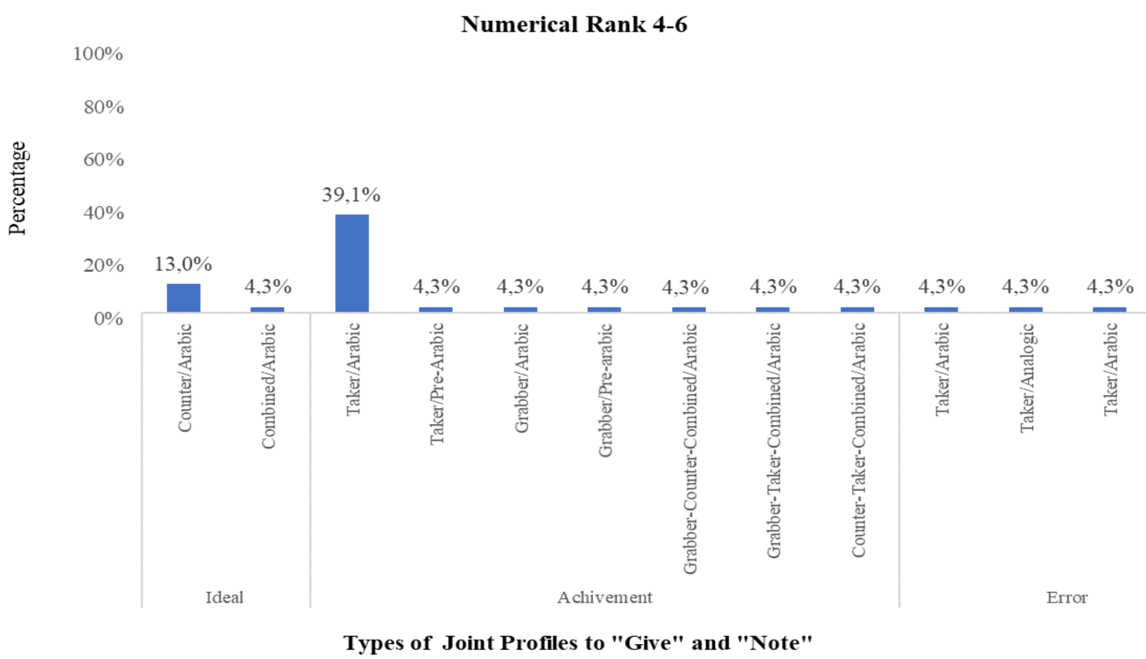

Figure 6. Percentage of joint profiles of cardinality-notation for the numerical rank 4-6.

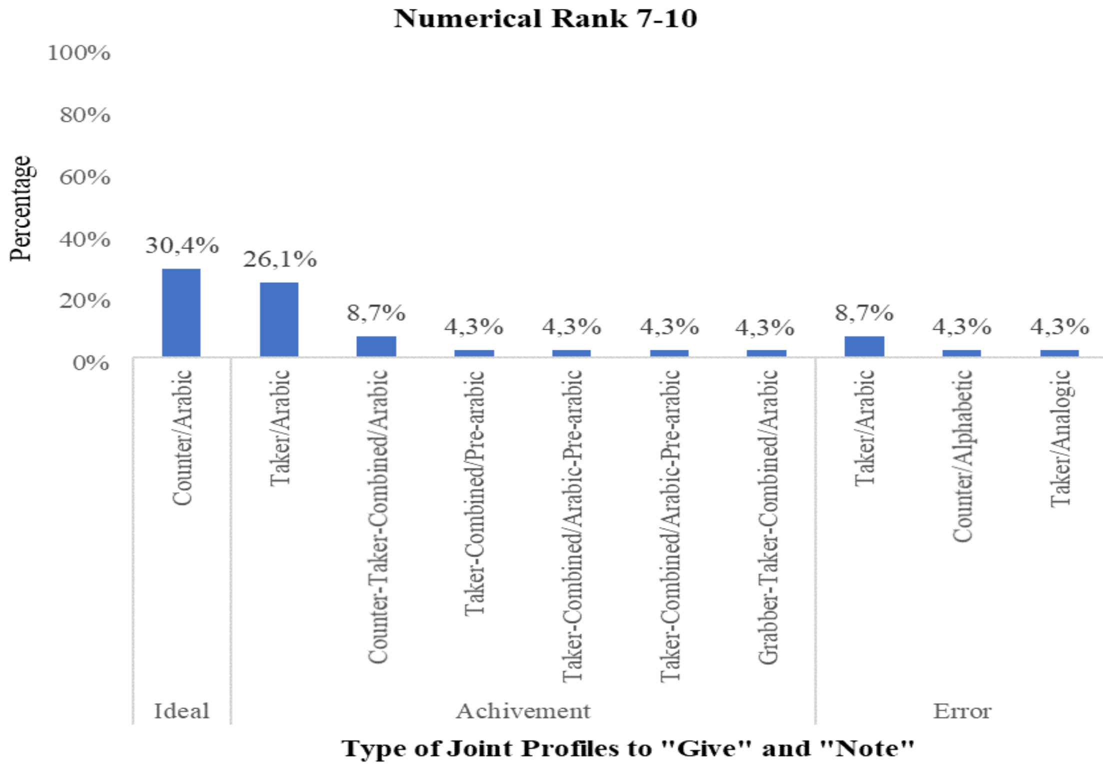

Figure 7. Percentage of joint profiles of cardinality-notation for the numerical rank 4-6

only children who are "accountants" can give a correct answer to a cardinal. As a result, this type of performance has been considered a hallmark of cardinal acquisition. Similarly, LeCorre y Carey (2007) suggest levels of numerical knowledge until reaching a successful acquisition of the cardinal.

Also, traditional studies about notations present a linear approach to knowledge development by conceiving stages linked to chronologic age. For instance, Munn (2001), suggests that children acquire the symbolic function when they used arabic numbers to represent the quantity of a set. In addition, Scheuer, et al. (2010), suggest that the notation is acquired when children represent a quantity by using Arabic numbers. In contrast, from an approach that we could consider nonlinear, Chentland and Fluck (2007) reveal that cardinal knowledge requires that children use diverse strategies that reveal their understanding of the numerical cardinal. These authors conceive that the "counter" strategy is not the only path to consider that a child has a cardinal understanding. For instance, the authors provide evidence about how a variety of strategies with different levels of complexity ("grabber", "taker", "combination") involves children's comprehension of the numerical cardinal. As a result, Chentland and Fluck offer new insights into the study of numerical knowledge by focusing on the variability in children's performance. The authors point out that children's strategies do not follow alinear sequence of development of numerical knowledge, but a coexistence.

The variability in the performance patterns observed in our study is related to the Overlapping Waves model of Robert Siegler (2000, 2006). It is because the intra-individual trajectories of children´s performance on the task show the simultaneous presence of different levels of complexity in their actions to establish cardinality and to note quantities. In terms of Siegler, this represents forward and backward, in short periods between weekly sessions. For instance, regarding the cardinality, for the set size 4 -6, children mainly use and strategy of low complexity "Grabber -Arabic" but also, they used a strategy of high complexity "Taker- Arabic") on the other three set of quantities 1-3, 4-6, and 7-10. These results indicate that children's strategies of cardinality are articulated with the conventional notation of quantities regardless of their level of complexity.

Taking an intra-individual approach to the study of numerical comprehension, the present study examined the joint use of cardinality and notation. Our findings reveal the 4-to 5-year-old children solving the task “Give and Note a Number”, show a nonlinear trajectory in their performance, by using a variety of strategies and notations along the three set sizes (1-3, 4-6, 7-10).This finding is related to the results of Chentland and Fluck (2007), who found a variety of strategies used within and between sessions. These authors demonstrate the use of “combination” as a new and complex strategy of cardinality, which has not been previously reported in the literature. We also evidence the use of that strategy for all the numerical ranks.

In our study, the children’s solving the task “Give and note” shows variability, but also patterns or regularities such as five clusters of intra-individual trajectories for both, to “give” and “note”, and diverse profiles of the joint use of the numerical skills of cardinality and notation. From a nonlinear approach, the numerical understanding of cardinal and notation is conceived as a process of knowledge in which children's performances go back and forth, varying the complexity of these numerical skills.

In contrast to traditional studies on numerical knowledge in which ideal performance is the target (i.e., “counter” strategy and “arabic notations”), the findings of our study go beyond the “ideal” or“success” performance. First, we provide an integrative view of cardinality and notation. Second, based on the identification of profiles, ourfindings offer insights into the joint use of cardinality and notations. And third, the intra- individual analysis of children’s trajectories showed the coexistence of diverse complexity of actions to “give” and “note”.

The joint profiles of cardinality-notation that we identify, indicate different paths of the children’s comprehension of the numerical task “Give and Note a Number”, such as combining lowcomplex strategies of cardinality (e.g., “Grabber”, “Taker strategy”), intermedium complex strategies (i.e., “Taker”) and high complex strategies (e.g., “Counter” and “Combined”) with a canonical arabic notation. The diversity of these profiles shows that although a child can use an arabic notation, it does not mean that their cardinality must correspond to a unique ideal strategy of cardinality (e.g., “counter” strategy to give). On the contrary, an ideal “arabic notation” can appear aside from diverse strategies to “give”that also allows achieving the task solution. Inaddition, the joint profiles of cardinality-notation depict an overlapping dynamic among the set size. The resulting variations are consistent with the Overlapping Waves Model proposed by Siegler (2000).

According to our results on a short-time scale, we conclude that children’s numerical skills of cardinality and notation, are dynamic and nonlinear. This evidence goes against the conception in which the numerical cardinal is based on the representation of the “word number”as the main criterion or predictor of numerical knowledge in formal schooling (Chu, VanMarle &Geary, 2015, Geary & VanMarle, 2018, Geary, VanMarle, Chu, Hoard & Nugent, 2019).

Finally, the recommendations for future research consist of the following suggestions: 1) to use a bigger sample size following an intra-individual analysis to gain more insights about how children use cardinality and use strategies of notations. 2) A bigger sample size can provide more information about the intra-individual patterns, revealing if there are more patterns or if the identified patterns of this study are more frequent than others or more stable over time. 3) to compare the intra-individual trajectories of different samples to explore if there are new or different joint profiles and patterns of cardinality and notations. These suggestions will provide more insights into the self-organization process of the children’s numerical comprehension of an integrative view of the skills of cardinality and notations.

6. Conflict of Interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

7. Author Contributions

All authors critically reviewed the manuscript and gave final approval of the version for publication.

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